Perfectoid Spaces

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$\boxed{\text{Recall}}$: (A, A^+) affinoid ring / Huber pair (i.e. A f-adic, $A^+ \underset{\text{open}}{\subseteq} A$ integrally closed, power-bounded)

$\rightsquigarrow \mathrm{Spa}(A,A^+)$: points are continuous valuations, valuing A^+ to almost 1

$\rightsquigarrow$ topology: $R\left(\frac{f_1,\dots,f_n}{g}\right) = \{ |\cdot| \in \mathrm{Spa}(A,A^+) \mid \forall i: |f_i| \leq |g| \neq 0 \}$, where $(f_1,\dots,f_n)_A$ is an open ideal in A

$\rightsquigarrow$ towards structure sheaf: localization $A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle$

$\hookrightarrow$ Goal: $\mathrm{Spa}\left(A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, ?\right) \cong R\left(\frac{f_1,\dots,f_n}{g}\right)$

$$\overset{\text{\rotatebox{90}{$\cong$}}}{\mathrm{Spa}\left(A\left(\frac{f_1,\dots,f_n}{g}\right), ?\right)}$$

Recall also: $A\left(\frac{f_1,\dots,f_n}{g}\right) \underset{\substack{\text{as a}\\ \text{ring}}}{\cong} A[g^{-1}] = A_g$, topology: "subring gen'd by $\frac{f_i}{g}$" $\cdot I^n$ is a fund. system (I defining ideal)

$\hookrightarrow A\left(\frac{f_1,\dots,f_n}{g}\right)^+ = \overline{A^+\left[\frac{f_i}{g} \,\middle|\, i=1,\dots,n\right]}$ integral closure inside $A\left(\frac{f_1,\dots,f_n}{g}\right)$ ($A^+\left[\frac{f_i}{g} \,\middle|\, i=1,\dots,n\right]$ isn't necessarily integrally closed).

$\boxed{\text{Key Proposition}}$: A f-adic, $\Sigma \subseteq A$ subset. $A_\Sigma :=$ smallest int'ly closed open subring of A containing Σ (in particular, this exists).

Then: $A_\Sigma = \{ a \in A \mid \forall |\cdot| \in \mathrm{Spa}(A, \Sigma): |a| \leq 1 \}$,

where $\mathrm{Spa}(A,\Sigma) := \{ |\cdot| \in \mathrm{Cont}(A) \mid |\Sigma| \leq 1 \}$.

$\boxed{\text{Pf}}$: (A_0, I) couple of def of A.

$\boxed{1}$ A_Σ exists: $\Sigma \subseteq B_i \underset{\substack{\text{open}\\ \text{subring}}}{\overset{\substack{\text{int'ly}\\ \text{closed in } A}}{\subseteq}} A$, for some index set $i \in \mathcal{B}$

want: $\bigcap_{i \in \mathcal{B}} B_i$ also such an object. problem: openness

Fix $i \in \Omega$: B_i open $\Rightarrow \exists j: I^j \subseteq B_i \overset{B_i \text{ int'ly closed}}{\Longrightarrow} I \subseteq B_i \Rightarrow \bigcap_{i \in \mathcal{B}} B_i \supseteq I$

$\hookrightarrow \bigcap_{i \in \mathcal{B}} B_i$ open. $\rightarrow A_\Sigma$ exists.

We also showed that $A_\Sigma \supseteq I$.

Now define $\tilde{A} = \{ a \in A \mid \forall |\cdot| \in \mathrm{Spa}(A,\Sigma): |a| \leq 1 \}$

$\boxed{2}$ Thoughts about $\tilde{A}$:

$\boxed{\text{i}}$ $\tilde{A} \subseteq A$ subring

$\boxed{\text{ii}}$ $I \subseteq \tilde{A}$: Take $|\cdot| \in \mathrm{Cont}(A)$, $\gamma \in \Gamma_{|\cdot|} = \Gamma \Rightarrow \{ a \in A \mid |a| < \gamma \}$ open $\Rightarrow$ contains I^t for some t.

$\leadsto$ choose $\gamma = 1 \Rightarrow \forall a \in I : |a^{+}| < 1 \Rightarrow |I| < 1$

$\boxed{\text{iii}}$ $\underline{\tilde{A} \text{ open}}$ (by $\boxed{\text{ii}}$)

$\boxed{\text{iv}}$ $\underline{\tilde{A} \text{ int'ly closed in } A}$: Take $f \in A$ s.t. $f^n + \sum_{i=0}^{n-1} a_i f^i = 0$, where $a_i \in \tilde{A}$.

For $|\cdot| \in \mathrm{Spa}(A, \Sigma')$, we $\overset{\text{then}}{}$ have $|f^n| \leq \max\{ |f|^i \mid i = 0, .., n-1 \} \Rightarrow |f| \leq 1$.

$\boxed{\text{v}}$ $\underline{\tilde{A} \supseteq A_{\Sigma'}}$ (by definition of $A_{\Sigma'}$ and the above)

$\boxed{3}$ $\underline{\tilde{A} = A_{\Sigma'}}$: Fix some [illegible] $a \notin A_{\Sigma}$.

$\hookrightarrow$ $\underline{\text{enough}}$: $\exists\, |\cdot| \in \mathrm{Spa}(A, \Sigma')$: $|a| > 1$. $\underline{\text{Strategy}}$: localize at a and construct $|\cdot|$ with $|a^{-1}| < 1$.

$A_{\Sigma'}[a^{-1}] \subseteq A[a^{-1}]$ are not the zero ring, as a not nilpotent

$\underline{\text{Want}}$: a^{-1} is contained in a prime ideal of $A_{\Sigma'}[a^{-1}]$

$\hookrightarrow$ $\underline{\text{enough}}$: $a \notin A_{\Sigma'}[a^{-1}]$

$\underline{\text{Pf}}$: Suppose by contradiction $a \in A_{\Sigma'}[a^{-1}] \Rightarrow a = \sum_{i=0}^{s} \frac{b_i}{a^i}$ in $A[a^{-1}]$ (where $b_i \in A_{\Sigma'}$)

$\Rightarrow a^{s+1} = \sum_{i=0}^{s} b_{s-i}\, a^i$ in $A[a^{-1}]$ $\Rightarrow \exists t > 0 : a^{t+s+1} = \sum_{i=0}^{s} b_{s-i}\, a^{i+t}$ $\underline{\text{in } A}$

$\Rightarrow a$ integral over $A_{\Sigma} \Rightarrow a \in A_{\Sigma'}$ $\lightning$ $\square$ (Pf($\underline{\text{enough}}$))

$\Rightarrow \exists\, \mathfrak{p} \subseteq A_{\Sigma'}[a^{-1}]$ a prime ideal with $a^{-1} \in \mathfrak{p}$.

Choose $\mathfrak{q} \subseteq \mathfrak{p}$ minimal prime of $A_{\Sigma'}[a^{-1}]$. $A_{\Sigma'}[a^{-1}] \to \overset{\mathfrak{p}/\mathfrak{q}}{\overset{\rotatebox{45}{$\subseteq$}}{A_{\Sigma'}[a^{-1}]/\mathfrak{q}}} \hookrightarrow \mathrm{Frac}(A_{\Sigma'}[a^{-1}]/\mathfrak{q})$

$(A_{\Sigma'}[a^{-1}]/\mathfrak{q}, \mathfrak{p}/\mathfrak{q})$ is dominated by a valuation ring in $\mathrm{Frac}(A_{\Sigma'}[a^{-1}]/\mathfrak{q})$

$\hookrightarrow \exists\, |\cdot|_{A_{\Sigma'}[a^{-1}]}$ s.t. $|A_{\Sigma'}[a^{-1}]| \leq 1$, $|a^{-1}| < 1$.

$$\underbrace{(A_{\Sigma'}[a^{-1}] \setminus \mathfrak{q})^{-1} A_{\Sigma'}[a^{-1}]}_{\neq 0} \underset{\Longrightarrow}{\hookrightarrow} \underbrace{(A_{\Sigma'}[a^{-1}] \setminus \mathfrak{q})^{-1} A[a^{-1}]}_{\neq 0}$$

$\hookrightarrow \exists\, \tilde{\mathfrak{q}} \subseteq A[a^{-1}]$ prime s.t. $\tilde{\mathfrak{q}} \cap A_{\Sigma'}[a^{-1}] \subseteq \mathfrak{q} \overset{\mathfrak{q} \text{ minimal}}{\Longrightarrow} \tilde{\mathfrak{q}} \cap A_{\Sigma'}[a^{-1}] = \mathfrak{q}$

$\hookrightarrow$

$$\begin{tikzcd}
 & A_{\Sigma'}[a^{-1}] \arrow[r] \arrow[d, hook] & A_{\Sigma'}[a^{-1}]/\mathfrak{q} \arrow[r] \arrow[d, hook] & \mathrm{Frac}(A_{\Sigma'}[a^{-1}]/\mathfrak{q}) \arrow[d, hook] \\
A \arrow[r] & A[a^{-1}] \arrow[r] & A[a^{-1}]/\mathfrak{q} \arrow[r] & \mathrm{Frac}(A[a^{-1}]/\tilde{\mathfrak{q}})
\end{tikzcd}$$

$\Rightarrow$ get $|\cdot|_{A[a^{-1}]}$ extending $|\cdot|_{A_{\Sigma'}[a^{-1}]}$ ($\Rightarrow |a|_{A[a^{-1}]}\, |a^{-1}|_{A[a^{-1}]} = 1 \Rightarrow$ they are not 0).

$\Rightarrow$ get $|\cdot|_A$ s.t. $|A_{\Sigma}|_A \leq 1$, $|a|_A > 1$.

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We have: $|\cdot|_A : A \longrightarrow \Gamma_A$

Want: $\forall \gamma \in \Gamma_A : \{a \in A \mid |a|_A < \gamma\}$ open

First: $\gamma = 1 \rightsquigarrow$ **Need:** $\exists t > 0 : |I^t|_A < 1$

Pf: By continuity of mult. by a:

$\exists t > 0 : I^t \cdot a \subseteq I \subseteq A_{\geq\gamma} \Rightarrow |I^t a|_A \leq 1 \overset{|a|_A > 1}{\Rightarrow} |I^t|_A < 1$

If we had: $b \in A : |b|_A > \gamma^{-1}$, we would get by replacing a with b that $|I^t| < \gamma$.

Bad news: not possible in general

↳**Ex:** $\mathbb{Z}_p\langle x\rangle$, we saw valuations s.t. $|\mathbb{Z}_p\langle x\rangle| \leq 1$.

$\boxed{\text{Def}}$: [① $|\cdot| : A \longrightarrow \Gamma$, $H \subseteq \Gamma$ (convex) subgroup. Define $|\cdot|_H : A \longrightarrow H$ by

$|a|_H := \begin{cases} |a| & \text{if } |a| \in H \\ 0 & \text{else} \end{cases}$ **Warning:** might not be a valuation!]

[② $c\Gamma_{|\cdot|}$ **characteristic (sub)group** is the convex subgroup of Γ gen'd by $|A|_{\geq 1}$]

Claim: $c\Gamma_{|\cdot|}$ is described by: $\gamma \in c\Gamma_{|\cdot|} \Leftrightarrow \exists m < 0 < n$ integers and $\exists a \in A$ with $|a| > 1$ s.t. $|a|^m < \gamma < |a|^n$

Pf (Claim): next time.

$\boxed{\text{Prop}}$ $H \subseteq \Gamma$ convex subgroup, TFAE:

(i) $|\cdot|_H$ is a valuation

(ii) $c\Gamma_{|\cdot|} \subseteq H$

So, returning to the original proof, if we put $|\cdot| = |\cdot|_A\big|_{c\Gamma_{|\cdot|_A}}$, then

$\boxed{i}$ $|\cdot|$ is continuous, $\boxed{ii}$ $|a| = |a|_A$ because $|a| > 1 \Rightarrow |a| \in c\Gamma_{|\cdot|_A}$

$\boxed{iii}$ $|A_{\geq\gamma}| \leq |A_{\geq\gamma}|_A \leq 1$.